

Calculus III: Coursework 9
M.A.H. MacCallum 2006

This coursework is optional. That is why it is numbered out of sequence.

The deadline is 5PM on Tuesday November 28th. Hand-in is to the green box in the basement of the Maths building.

1. Show that over any closed surface $\mathcal{S}$

$$\int_{\mathcal{S}} \mathbf{r} \cdot d\mathbf{S} = 3 \times \text{enclosed volume}.$$

Hence explain the answers to Coursework 4, question 4.

[12]

Answer

By the Divergence Theorem, $\int_{\mathcal{S}} \mathbf{r} \cdot d\mathbf{S} = \int_{\mathcal{D}} \nabla \cdot \mathbf{r} dV$ where $\mathcal{D}$ is the enclosed region.

$\nabla \cdot \mathbf{r} = 3$ so this is $\int_{\mathcal{D}} 3 dV = 3V$ where V is the volume enclosed.

In Coursework 4, Q4, you were asked to calculate the surface integral directly for two volumes, a rectangular box with sides $2a$, $2b$ and $2c$ and hence of volume $8abc$, and a sphere of radius a and volume $\frac{4}{3}\pi a^3$. Using the divergence theorem the results are examples of the calculation just done.

Marks: 4 for correct use of Div. Thm, 4 for first part, 4 for correct comments on old coursework

2. Using the Divergence Theorem, evaluate $\int_{\mathcal{S}} \mathbf{A} \cdot \mathbf{n} dS$ where $\mathbf{A} = (x+z)\mathbf{i} + (y+z)\mathbf{j} + (x+y)\mathbf{k}$ and $\mathcal{S}$ is the surface of the region $0 \leq x \leq 2$, $0 \leq y^2 + z^2 \leq 1$. [If feeling strong, you might like to try doing this by direct integration over the surface, to see how much more awkward it is!]

[16]

Answer The divergence of $\mathbf{A}$ is

$$\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y) = 2$$

and the region is a cylinder with circular base of radius 1 in the (y, z) plane and length 2. Its volume is therefore $\pi \cdot 1^2 \cdot 2 = 2\pi$. So the answer is 4π .

Marks 4 for correct use of Div Thm, 4 for correct divergence, 4 for identification of volume (in a sketch or words), 4 for final result.

3. An important theorem in complex variables (called Cauchy's theorem) states that if there are two well-behaved functions $u(x, y)$ and $v(x, y)$ satisfying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and an arbitrary closed curve C in the x, y plane then

$$\oint_C (u + iv)(dx + idy) = 0.$$

Show that Cauchy's theorem is a special case of Green's theorem in the plane.

Hint: Try separating out the real and imaginary parts of the integral. [12]

Answer

The real and imaginary parts of the integral are

$$\int (u \, dx - v \, dy) + i \int (v \, dx + u \, dy) .$$

Applying Green's theorem gives

$$- \iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx \, dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \, dy .$$

The given relation between the derivatives makes both integrands zero.

Marks: Separating real and imaginary parts (4), applying Green's theorem (4), simplifying (4).

Comments In complex analysis one would write $f(z) = u + iv$ where $z = x + iy$ and the result is $\int_C f \, dz = 0$.

4. Using Stokes's Theorem, evaluate $\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, dS$ for $\mathbf{A} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and the surface S given by $z \geq 0$, $x^2 + y^2 + z^2 = a^2$. [16]

Answer

The curve C bounding S is the circle $x^2 + y^2 = a^2$. Round it we can use a parameter θ such that $x = a \cos \theta$, $y = a \sin \theta$ (polar coordinates, in fact).

Using Stokes's theorem $\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, dS = \int_C \mathbf{A} \cdot d\mathbf{r}$. The line integral involves

$$\begin{aligned} d\mathbf{r} &= a(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) d\theta, \\ \mathbf{A} &= a \cos \theta \mathbf{j} + a \sin \theta \mathbf{k} \end{aligned}$$

round the curve, leaving an integral $a^2 \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi a^2$.

Marks: correct use of Stokes's theorem 4, parametrizing C correctly 4, correct integral formed 4, final answer 4.

Note: a less likely but also correct argument would be to note that the answer must, by Stokes's theorem, be the same for the flat disc in the (x, y) plane, $\nabla \times \mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $d\mathbf{S} = \mathbf{k} \, dx \, dy$ so we have $\int \int dx \, dy$ over a circle which gives same result. Divide the marks in a similar manner.

5. Explain the answers to Coursework 7, question 7, using vector integral theorems. [8]

Answer

The line integral of $\mathbf{F}$ in Coursework 7, question 7 is equal to the surface integral of $\mathbf{G} = \nabla \times \mathbf{F}$ over any surface spanning the curve in part (a), by Stokes's theorem. Parts (b) and (c) are examples of such surfaces. The relation between parts (a) and (b) can also be seen as an example of Green's theorem in the plane.

Marks 6 for correct explanation, 2 for accuracy in expressing it.

6. Show that the following are all zero.

- (a) $\oint \mathbf{r} \cdot d\mathbf{r}$ over an arbitrary closed path.
- (b) $\oint \mathbf{r} \times d\mathbf{S}$ over an arbitrary closed surface. [*Hint*: Try taking the $\mathbf{k}$ component.]
- (c) $\oint d\mathbf{S}$ over an arbitrary closed surface.

[36]

Answer

- (a) Using Stokes theorem

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{r}) \cdot d\mathbf{S} = 0$$

- (b) Consider the $\mathbf{k}$ component:

$$\mathbf{k} \cdot \oint \mathbf{r} \times d\mathbf{S} = \oint (\mathbf{k} \times \mathbf{r}) \cdot d\mathbf{S} = \int \nabla \cdot (\mathbf{k} \times \mathbf{r}) dV$$

by the divergence theorem. The divergence evaluates to zero, hence the integral is zero. This applies similarly for the $\mathbf{i}$ and $\mathbf{j}$ components so the original vector is zero.

- (c) Again consider the $\mathbf{k}$ component:

$$\mathbf{k} \cdot \oint d\mathbf{S} = \int (\nabla \cdot \mathbf{k}) dV = 0$$

Complete by the same argument as the previous part.

Marks: (a) Invoking Stokes' theorem correctly 4, realizing and using that $\nabla \times \mathbf{r} = 0$, 4; (b) taking component correctly 2, rewriting triple product 4, using divergence theorem 4, showing divergence is zero 4, final step 4; (c) again using components 6 (4+2 for last step), using divergence theorem 4.